

ROTATIONAL SYMMETRY OF SELF-SIMILAR SOLUTIONS TO THE RICCI FLOW

SIMON BRENDLE

ABSTRACT. Let (M, g) be a three-dimensional steady gradient Ricci soliton which is κ -noncollapsed and non-flat. We prove that (M, g) is isometric to the Bryant soliton up to scaling. This answers a question mentioned in Perelman's first paper [18].

1. INTRODUCTION

Self-similar solutions play a central role in the study of the Ricci flow, and have been studied extensively in connection with singularity formation; see e.g. the work of R. Hamilton [12] and G. Perelman [18], [19], [20]. There are three basic types of self-similar solutions, which are referred to as shrinking solitons; steady solitons; and expanding solitons. A steady Ricci soliton (M, g) is characterized by the fact that $\text{Ric} = \frac{1}{2} \mathcal{L}_X(g)$ for some vector field X . If the vector field X is the gradient of a function, we say that (M, g) is a steady gradient Ricci soliton.

The simplest example of a steady Ricci soliton is the cigar soliton in dimension 2, which was found by Hamilton (cf. [12]). R. Bryant [3] has discovered a steady Ricci soliton in dimension 3, which is rotationally symmetric. Moreover, Bryant showed that there are no other steady Ricci solitons in dimension 3 which are rotationally symmetric. While additional examples are known in higher dimensions (see e.g. [16]), the Bryant soliton is the only known steady Ricci soliton in dimension 3. It is an interesting question whether any three-dimensional steady Ricci soliton is necessarily rotationally symmetric. Perelman mentions this problem in his first paper (see [18], p. 32), but does not indicate an outline of a possible proof. We note that several authors have obtained uniqueness results for the Bryant soliton and its higher dimensional counterparts under various additional assumptions. We refer to [4], [5], [6], and [8] for details.

In this paper, we prove the uniqueness of the Bryant soliton under a noncollapsing assumption:

Theorem 1.1. *Let (M, g) be a three-dimensional complete steady gradient Ricci soliton which is κ -noncollapsed and non-flat. Then (M, g) is rotationally symmetric, and is therefore isometric to the Bryant soliton up to scaling.*

We now outline the main steps involved in the proof of Theorem 1.1. We will assume throughout that (M, g) is a three-dimensional complete steady gradient Ricci soliton which is κ -noncollapsed and non-flat. It follows from Theorem 1.3 in [26] that (M, g) has positive scalar curvature. We may write $\text{Ric} = D^2 f$ for some real-valued function f . It is well known that the sum $R + |\nabla f|^2$ is constant. By scaling, we may assume that $R + |\nabla f|^2 = 1$. For abbreviation, we put $X = \nabla f$. Moreover, we denote by Φ_t the one-parameter group of diffeomorphisms generated by the vector field $-X$. Since $|\nabla f| \leq 1$, the flow Φ_t exists for all $t \in \mathbb{R}$, and the metrics $\Phi_t^*(g)$ evolve by the Ricci flow. Using the local version of the Hamilton-Ivey pinching estimate established by B.L. Chen, we conclude that (M, g) has nonnegative sectional curvature (see [7], Corollary 2.4). Since $R \leq 1$, it follows that (M, g) has bounded curvature. Since (M, g) is κ -noncollapsed and has positive scalar curvature, (M, g) cannot split off a line. Consequently, (M, g) has strictly positive sectional curvature. In particular, the function f is strictly convex. Without loss of generality, we may assume that $\inf_M f \geq 1$.

In Section 2, we analyze the asymptotic geometry of (M, g) . It follows from work of Perelman [18] that the asymptotic shrinking soliton associated with (M, g) is a cylinder. This fact plays a fundamental role in our analysis. Moreover, we show that the restriction of the scalar curvature to the level surface $\{f = r\}$ satisfies $R = \frac{1}{r} + O(r^{-\frac{5}{4}})$. As a consequence, the intrinsic Gaussian curvature of the level surface $\{f = r\}$ equals $\frac{1}{2r} + O(r^{-\frac{5}{4}})$. This can be viewed as a roundness estimate for the level surface $\{f = r\}$.

In Section 3, we construct a collection of approximate Killing vector fields near infinity. More precisely, we construct three vector fields U_1, U_2, U_3 such that $|\mathcal{L}_{U_a}(g)| \leq O(r^{-2\varepsilon})$ and $|\Delta U_a + D_X U_a| \leq O(r^{-\frac{1}{2}-2\varepsilon})$ for some small number $\varepsilon > 0$. Moreover, we show that the vector fields U_1, U_2, U_3 satisfy $|U_a| \leq O(r^{\frac{1}{2}})$ and

$$\int_{\{f=r\}} \langle U_a, U_b \rangle = r^2 \delta_{ab} + O(r^{2-2\varepsilon})$$

if r is sufficiently large.

In Section 4, we consider a vector field W which satisfies the elliptic equation $\Delta W + D_X W = 0$. We then consider the Lie derivative $h = \mathcal{L}_W(g)$. We show that this tensor satisfies the Lichnerowicz-type equation

$$(1) \quad \Delta_L h + \mathcal{L}_X(h) = 0,$$

where Δ_L denotes the Lichnerowicz Laplacian, i.e.

$$\Delta_L h_{ik} = \Delta h_{ik} + 2 R_{ijkl} h^{jl} - \text{Ric}_i^l h_{kl} - \text{Ric}_k^l h_{il}.$$

In Section 5, we assume that a vector field Q satisfying $|Q| \leq O(r^{-\frac{1}{2}-2\varepsilon})$ is given. We then construct a vector field V such that $\Delta V + D_X V = Q$ and $|V| \leq O(r^{-\frac{1}{2}-\varepsilon})$. In order to construct the vector field V , we solve the Dirichlet problem on a sequence of domains which exhaust M . In order to be able to pass to the limit, we need uniform estimates for solutions of the

equation $\Delta V + D_X V = Q$. These estimates are established using a delicate blow-down analysis; see Propositions 5.2 and 5.3 below.

In Section 6, we consider a symmetric $(0, 2)$ -tensor h which solves the equation (1) and satisfies $|h| \leq O(r^{-\varepsilon})$ at infinity. Note that such a tensor h need not vanish identically. Indeed, the Ricci tensor of (M, g) is a non-trivial solution of the equation (1), which falls off like r^{-1} at infinity. However, we are able to show that any solution of (1) with $|h| \leq O(r^{-\varepsilon})$ is of the form $h = \lambda \text{Ric}$ for some constant $\lambda \in \mathbb{R}$; see Theorem 6.2 below. The proof of Theorem 6.2 again relies on a blow-down argument. We also use an inequality due to G. Anderson and B. Chow [1] for solutions of the parabolic Lichnerowicz equation. Related ideas were used in earlier work of M. Gursky [10] and R. Hamilton [11].

Finally, in Section 7, we establish a crucial symmetry principle. To explain this, suppose that U is a vector field on (M, g) such that $|\mathcal{L}_U(g)| \leq O(r^{-2\varepsilon})$ and $|\Delta U + D_X U| \leq O(r^{-\frac{1}{2}-2\varepsilon})$ for some small constant $\varepsilon > 0$. Using the results in Section 5, we can find a vector field V such that $\Delta V + D_X V = \Delta U + D_X U$ and $|V| \leq O(r^{-\frac{1}{2}-\varepsilon})$. Therefore, the vector field $W = U - V$ satisfies $\Delta W + D_X W = 0$. Consequently, the Lie derivative $h = \mathcal{L}_W(g)$ is a solution of the equation (1). Moreover, we can show that $|h| \leq O(r^{-\varepsilon})$ at infinity. Thus, $h = \lambda \text{Ric}$ for some constant $\lambda \in \mathbb{R}$. From this, we deduce that the vector field $\hat{U} := W - \frac{1}{2} \lambda X$ is a Killing vector field. Moreover, the Killing vector field \hat{U} agrees with the original vector field U up to terms of order $O(r^{\frac{1}{2}-\varepsilon})$. Applying this symmetry principle to the approximate Killing vector fields U_1, U_2, U_3 constructed in Section 3, we obtain three exact Killing vector fields $\hat{U}_1, \hat{U}_2, \hat{U}_3$ on (M, g) satisfying $|\hat{U}_a| \leq O(r^{\frac{1}{2}})$ and

$$\int_{\{f=r\}} \langle \hat{U}_a, \hat{U}_b \rangle = r^2 \delta_{ab} + O(r^{2-\varepsilon})$$

if r is sufficiently large.

Finally, we mention some related results. Our method of proof is inspired in part by the beautiful work of L. Simon and B. Solomon on the uniqueness of minimal hypersurfaces in \mathbb{R}^{n+1} which are asymptotic to a given cone at infinity (cf. [21], [22]). X.J. Wang [25] has obtained a uniqueness theorem for convex translating solutions to the mean curvature flow in \mathbb{R}^3 . Finally, the uniqueness problem for the Bryant soliton shares some common features with the black hole uniqueness theorems in general relativity (see e.g. [13], [14], [15]).

It is a pleasure to thank Professors Huai-Dong Cao, Gerhard Huisken, Sergiu Klainerman, Leon Simon, Brian White, and Meng Zhu for discussions. The author is grateful to the Department of Mathematics at Princeton University, where part of this work was carried out.

2. THE ASYMPTOTIC GEOMETRY OF (M, g)

In this section, we analyze the asymptotic geometry of (M, g) near infinity. The following result was established by H. Guo [9]:

Proposition 2.1 (H. Guo [9]). *Fix a point $x_0 \in M$. The function f satisfies $f(x) = (1 + o(1)) d(x_0, x)$ as $x \rightarrow \infty$. Moreover, we have $\inf_M f R > 0$ and $\sup_M f R < \infty$.*

The following theorem was established by Perelman:

Theorem 2.2 (G. Perelman [18], [19]). *Consider a sequence of real numbers $r_m \rightarrow \infty$ and a sequence of marked points p_m such that $f(p_m) = r_m$. Moreover, let*

$$\hat{g}^{(m)}(t) = r_m^{-1} \Phi_{r_m t}^*(g).$$

As $m \rightarrow \infty$, the flows $(M, \hat{g}^{(m)}(t), p_m)$ converge in the Cheeger-Gromov sense to a family of shrinking cylinders $(S^2 \times \mathbb{R}, \bar{g}(t))$, $t \in (0, 1)$. The metric $\bar{g}(t)$ is given by

$$(2) \quad \bar{g}(t) = (2 - 2t) g_{S^2} + dz \otimes dz,$$

where g_{S^2} denotes the standard metric on S^2 with constant Gaussian curvature 1. Furthermore, the rescaled vector fields $r_m^{\frac{1}{2}} X$ converge to the axial vector field $\frac{\partial}{\partial z}$ on $S^2 \times \mathbb{R}$.

Proof. It follows from Proposition 11.2 in [18] and Lemma 1.2 in [19] that the flows $(M, \hat{g}^{(m)}(t), p_m)$ converge in the Cheeger-Gromov sense to a family of round cylinders $(S^2 \times \mathbb{R}, \bar{g}(t))$, $t \in (0, 1)$, which evolve by the Ricci flow. By Proposition 2.1, the scalar curvature of $\bar{g}(t)$ satisfies

$$\frac{c_1}{1-t} \leq R_{\bar{g}(t)} \leq \frac{c_2}{1-t}$$

for all $t \in (0, 1)$, where c_1 and c_2 are positive constants. From this, we deduce that $R_{\bar{g}(t)} = \frac{1}{1-t}$. This shows that

$$\bar{g}(t) = (2 - 2t) g_{S^2} + dz \otimes dz,$$

where g_{S^2} denotes the standard metric on S^2 with constant Gaussian curvature 1. Finally, the rescaled vector fields $r_m^{\frac{1}{2}} X$ converge to a smooth vector field \bar{X} on $S^2 \times \mathbb{R}$. It is easy to see that the vector field \bar{X} is parallel with respect to the metric $\bar{g}(t)$ for each $t \in (0, 1)$. Moreover, $|\bar{X}|_{\bar{g}(t)} = 1$ for each $t \in (0, 1)$. Thus, \bar{X} can be identified with the axial vector field $\frac{\partial}{\partial z}$ on $S^2 \times \mathbb{R}$.

Corollary 2.3. *We have $f R = 1 + o(1)$ as $x \rightarrow \infty$.*

In the remainder of this section, we establish a roundness estimate for the level surfaces $\{f = r\}$. The proof of this estimate requires several lemmata.

Lemma 2.4. *We have*

$$2 \operatorname{Ric}(\nabla f, \nabla f) = -\langle X, \nabla R \rangle = O(r^{-2}).$$

Proof. We have

$$2 \operatorname{Ric}(\nabla f, \nabla f) = -\langle X, \nabla R \rangle = \Delta R + 2 |\operatorname{Ric}|^2 = O(r^{-2}),$$

as claimed.

Lemma 2.5. *The mean curvature of the level surface $\{f = r\}$ equals $\frac{1+o(1)}{r}$.*

Proof. The mean curvature of the level surface $\{f = r\}$ is given by

$$H = \frac{1}{|\nabla f|} R - \frac{1}{|\nabla f|^3} \operatorname{Ric}(\nabla f, \nabla f).$$

Hence, the assertion follows from Corollary 2.3 and Lemma 2.4.

Lemma 2.6. *The tensor $T = 2 \operatorname{Ric} - Rg + Rdf \otimes df$ satisfies $|T| \leq O(r^{-\frac{3}{2}})$ and $|DT| \leq O(r^{-2})$.*

Proof. In dimension 3, the Riemann curvature tensor can be written in the form

$$\begin{aligned} R_{ijkl} &= \operatorname{Ric}_{ik} g_{jl} - \operatorname{Ric}_{il} g_{jk} - \operatorname{Ric}_{jk} g_{il} + \operatorname{Ric}_{jl} g_{ik} \\ &\quad - \frac{1}{2} R (g_{ik} g_{jl} - g_{il} g_{jk}). \end{aligned}$$

This implies

$$\begin{aligned} D_i \operatorname{Ric}_{jk} - D_j \operatorname{Ric}_{ik} &= R_{ijkl} D^l f \\ &= \operatorname{Ric}_{ik} D_j f - \operatorname{Ric}_{jk} D_i f \\ &\quad - \frac{1}{2} (D_j R + R D_j f) g_{ik} + \frac{1}{2} (D_i R + R D_i f) g_{jk}, \end{aligned}$$

hence

$$\begin{aligned} (3) \quad & 2 (D_i \operatorname{Ric}_{jk} - D_j \operatorname{Ric}_{ik}) D^j f \\ &= T_{ik} |\nabla f|^2 - \langle \nabla R, \nabla f \rangle g_{ik} + R^2 D_i f D_k f \\ &\quad + D_i R D_k f + D_k R D_i f. \end{aligned}$$

By Shi's estimate, the covariant derivatives of the curvature tensor are bounded by $O(r^{-\frac{3}{2}})$. Consequently, the identity (3) implies that $|T| \leq O(r^{-\frac{3}{2}})$. Moreover, if we differentiate (3), we obtain $|DT| \leq O(r^{-2})$.

Lemma 2.7. *We have*

$$|\langle X, \nabla R \rangle + \Delta_\Sigma R + R^2| \leq O(r^{-\frac{5}{2}}),$$

where Δ_Σ denotes the Laplacian on the level surface $\{f = r\}$.

Proof. Recall that

$$(4) \quad -\langle X, \nabla R \rangle = \Delta R + 2 |\operatorname{Ric}|^2.$$

Differentiating (4), we obtain

$$-(D^2 R)(X, X) - \langle D_X X, \nabla R \rangle = \langle X, \nabla(\Delta R + 2|\text{Ric}|^2) \rangle.$$

Since $\nabla R = -2D_X X$, it follows that

$$-(D^2 R)(X, X) = -\frac{1}{2}|\nabla R|^2 + \langle X, \nabla(\Delta R + 2|\text{Ric}|^2) \rangle.$$

Using Shi's estimates, we obtain $|\nabla R|^2 \leq O(r^{-3})$ and $|\nabla(\Delta R + 2|\text{Ric}|^2)| \leq O(r^{-\frac{5}{2}})$. Consequently, we have

$$(5) \quad |(D^2 R)(X, X)| \leq O(r^{-\frac{5}{2}}).$$

Moreover, it follows from Lemma 2.4 and Lemma 2.5 that

$$(6) \quad |H \langle X, \nabla R \rangle| \leq O(r^{-3}).$$

Combining (5) and (6) gives

$$|\Delta R - \Delta_\Sigma R| \leq O(r^{-\frac{5}{2}}).$$

Substituting this into (4), we obtain

$$|\Delta_\Sigma R + \langle X, \nabla R \rangle + 2|\text{Ric}|^2| \leq O(r^{-\frac{5}{2}}).$$

On the other hand, it follows from Lemma 2.6 that

$$2|\text{Ric}| = |R(g - df \otimes df)| + O(r^{-\frac{3}{2}}) = \sqrt{2}R + O(r^{-\frac{3}{2}}).$$

Putting these facts together, the assertion follows.

We next establish a Poincaré-type inequality for the restriction of the scalar curvature to a level surface $\{f = r\}$. Our argument uses the Kazdan-Warner identity (cf. [17]), and is inspired in part by work of M. Struwe on the Calabi flow on the two-sphere (cf. [23], p. 263). In the sequel, we denote by $\mu(r)$ the mean value of the scalar curvature over the level surface $\{f = r\}$, so that

$$\int_{\{f=r\}} (R - \mu(r)) = 0.$$

Note that $\mu(r) = \frac{1+o(1)}{r}$ by Corollary 2.3.

Lemma 2.8. *We have*

$$\int_{\{f=r\}} |\nabla^\Sigma R|^2 \geq \frac{2}{r} \left(\int_{\{f=r\}} (R - \mu(r))^2 \right) - O(r^{-4})$$

if r is sufficiently large.

Proof. Let us fix r sufficiently large. Let $0 = \nu_0 < \nu_1 \leq \nu_2 \leq \nu_3 \leq \dots$ denote the eigenvalues of the Laplace operator on the level surface $\{f = r\}$, and let $\psi_0, \psi_1, \psi_2, \psi_3, \dots$ denote the associated eigenfunctions. We assume that the eigenfunctions are normalized so that $\int_{\{f=r\}} \psi_j^2 = 1$ for each j . When r is large, the surface $\{f = r\}$ equipped with the rescaled metric $\frac{1}{2r}g$

is C^∞ close to the standard two-sphere with constant Gaussian curvature 1. Consequently, $\nu_1 = \frac{1+o(1)}{r}$, $\nu_2 = \frac{1+o(1)}{r}$, $\nu_3 = \frac{1+o(1)}{r}$, and $\nu_4 = \frac{3+o(1)}{r}$.

Let K denote the intrinsic Gaussian curvature of the level surface $\{f = r\}$. Using the Gauss equations, we obtain

$$2K = 2R(e_1, e_2, e_1, e_2) + O(r^{-2}) = R - \frac{2}{|\nabla f|^2} \text{Ric}(\nabla f, \nabla f) + O(r^{-2}).$$

Using Lemma 2.4, we conclude that $|2K - R| \leq O(r^{-2})$, hence

$$\left(\int_{\{f=r\}} (2K - R)^2 \right)^{\frac{1}{2}} \leq O(r^{-\frac{3}{2}}).$$

On the other hand, it follows from the Kazdan-Warner identity (see [17], Theorem 8.8) that

$$\sum_{j=1}^3 \left| \int_{\{f=r\}} (2K - \mu(r)) \psi_j \right| \leq o(1) \left(\int_{\{f=r\}} (2K - \mu(r))^2 \right)^{\frac{1}{2}}$$

Putting these facts together, we obtain

$$\sum_{j=1}^3 \left| \int_{\{f=r\}} (R - \mu(r)) \psi_j \right| \leq o(1) \left(\int_{\{f=r\}} (R - \mu(r))^2 \right)^{\frac{1}{2}} + O(r^{-\frac{3}{2}}).$$

Thus, we conclude that

$$\begin{aligned} & \int_{\{f=r\}} |\nabla^\Sigma R|^2 - \nu_4 \int_{\{f=r\}} (R - \mu(r))^2 \\ &= \sum_{j=1}^{\infty} (\nu_j - \nu_4) \left(\int_{\{f=r\}} (R - \mu(r)) \psi_j \right)^2 \\ &\geq -\frac{3}{r} \sum_{j=1}^3 \left(\int_{\{f=r\}} (R - \mu(r)) \psi_j \right)^2 \\ &\geq -o(r^{-1}) \left(\int_{\{f=r\}} (R - \mu(r))^2 \right) - O(r^{-4}). \end{aligned}$$

Since $\nu_4 = \frac{3+o(1)}{r}$, the assertion follows.

We now prove the crucial roundness estimate.

Proposition 2.9. *We have*

$$\int_{\{f=r\}} (R - \mu(r))^2 \leq O(r^{-2})$$

if r is sufficiently large.

Proof. By definition of $\mu(r)$, we have $\int_{\{f=r\}} (R - \mu(r)) = 0$. This implies

$$\begin{aligned}
& \frac{d}{dr} \left(\int_{\{f=r\}} (R - \mu(r))^2 \right) \\
&= 2 \int_{\{f=r\}} (R - \mu(r)) \left(\frac{\langle X, \nabla R \rangle}{|X|^2} - \mu'(r) \right) + \int_{\{f=r\}} H (R - \mu(r))^2 \\
&= 2 \int_{\{f=r\}} (R - \mu(r)) \left(\frac{\langle X, \nabla R \rangle}{|X|^2} + \mu(r)^2 \right) + \int_{\{f=r\}} H (R - \mu(r))^2 \\
&= 2 \int_{\{f=r\}} |\nabla^\Sigma R|^2 - \int_{\{f=r\}} (2R + 2\mu(r) - H) (R - \mu(r))^2 \\
&+ 2 \int_{\{f=r\}} (R - \mu(r)) \left(\frac{\langle X, \nabla R \rangle}{|X|^2} + \Delta_\Sigma R + R^2 \right).
\end{aligned}$$

It follows from Lemma 2.8 that

$$\int_{\{f=r\}} |\nabla^\Sigma R|^2 \geq \frac{2}{r} \left(\int_{\{f=r\}} (R - \mu(r))^2 \right) - O(r^{-4}).$$

Moreover, we have $2R + 2\mu(r) - H = \frac{3+o(1)}{r}$. Finally, we have

$$\left| \frac{\langle X, \nabla R \rangle}{|X|^2} + \Delta_\Sigma R + R^2 \right| \leq O(r^{-\frac{5}{2}})$$

by Lemma 2.7. Putting these facts together, we obtain

$$\begin{aligned}
\frac{d}{dr} \left(\int_{\{f=r\}} (R - \mu(r))^2 \right) &\geq \frac{1 - o(1)}{r} \int_{\{f=r\}} (R - \mu(r))^2 \\
&- O(r^{-\frac{5}{2}}) \int_{\{f=r\}} |R - \mu(r)| \\
&- O(r^{-4}).
\end{aligned}$$

Using Young's inequality, we conclude that

$$\begin{aligned}
\frac{d}{dr} \left(\int_{\{f=r\}} (R - \mu(r))^2 \right) &\geq -O(r^{-4}) \text{vol}(\{f = r\}) - O(r^{-4}) \\
&\geq -O(r^{-3}).
\end{aligned}$$

Clearly,

$$\int_{\{f=r\}} (R - \mu(r))^2 \rightarrow 0$$

as $r \rightarrow \infty$. Putting these facts together, we obtain

$$\int_{\{f=r\}} (R - \mu(r))^2 \leq O(r^{-2}),$$

as claimed.

Corollary 2.10. *We have*

$$\begin{aligned} \sup_{\{f=r\}} |R - \mu(r)| &\leq O(r^{-\frac{5}{4}}), \\ \sup_{\{f=r\}} |\nabla^\Sigma R| &\leq O(r^{-\frac{7}{4}}), \\ \sup_{\{f=r\}} |\Delta_\Sigma R| &\leq O(r^{-\frac{9}{4}}). \end{aligned}$$

Proof. By Proposition 2.9, we have

$$\int_{\{f=r\}} (R - \mu(r))^2 \leq O(r^{-2}).$$

Moreover, it follows from Shi's estimates that

$$\sup_{\{f=r\}} |D^l R| \leq C(l) r^{-\frac{l+2}{2}}.$$

Hence, the assertion follows from standard interpolation inequalities.

With the aid of Corollary 2.10, we can improve Corollary 2.3 as follows:

Proposition 2.11. *We have $|\nabla R| \leq O(r^{-\frac{7}{4}})$ and $fR = 1 + O(r^{-\frac{1}{4}})$.*

Proof. Using the estimates $|\nabla^\Sigma R| \leq O(r^{-\frac{7}{4}})$ and $|\langle X, \nabla R \rangle| \leq O(r^{-2})$, we obtain $|\nabla R| \leq O(r^{-\frac{7}{4}})$. This proves the first statement.

We now describe the proof of the second statement. By Corollary 2.10, we have $|\Delta_\Sigma R| \leq O(r^{-\frac{9}{4}})$. Hence, Lemma 2.7 implies

$$|\langle X, \nabla R \rangle + R^2| \leq O(r^{-\frac{9}{4}}).$$

From this, we deduce that

$$\left\langle X, \nabla \left(\frac{1}{R} - f \right) \right\rangle = -\frac{1}{R^2} \langle X, \nabla R \rangle - |\nabla f|^2 = 1 - |\nabla f|^2 + O(r^{-\frac{1}{4}}) = O(r^{-\frac{1}{4}}).$$

Integrating this relation along the integral curves of X gives

$$\frac{1}{R} - f = O(r^{\frac{3}{4}}).$$

From this, the assertion follows.

Corollary 2.12. *The mean curvature of the level surface $\{f = r\}$ is given by $\frac{1}{r} + O(r^{-\frac{5}{4}})$. Moreover, the intrinsic Gaussian curvature of the level surface $\{f = r\}$ is given by $\frac{1}{2r} + O(r^{-\frac{5}{4}})$.*

3. EXISTENCE OF APPROXIMATE KILLING VECTOR FIELDS NEAR INFINITY

In this section, we shall construct a collection of approximate Killing vector fields near infinity. Throughout this section, we fix a small number $\varepsilon > 0$. For example, $\varepsilon = \frac{1}{100}$ will work. Moreover, we define $\rho_m = 2^m$, $\Sigma_m = \{f = \rho_m\}$, and $\Omega_m = \{\rho_m \leq f \leq 4\rho_m\}$.

Proposition 3.1. *If m is large enough, we can find vector fields $U_1^{(m)}, U_2^{(m)}, U_3^{(m)}$ which are tangential to the surface Σ_m and satisfy the following properties:*

(i) *The covariant derivatives of $U_a^{(m)}$ satisfy*

$$\sup_{\Sigma_m} |D_{\Sigma}^l U_a^{(m)}| \leq C(l) \rho_m^{\frac{1-l}{2}}$$

for $a \in \{1, 2, 3\}$ and $l = 0, 1, 2, \dots$. Here, D_{Σ} denotes the Levi-Civita connection on Σ_m .

(ii) *If $\{e_1, e_2\}$ is a local orthonormal frame on Σ_m , then*

$$|\langle D_{e_i} U_a^{(m)}, e_j \rangle + \langle D_{e_j} U_a^{(m)}, e_i \rangle| \leq O(\rho_m^{-4\varepsilon}).$$

(iii) *We have*

$$\int_{\Sigma_m} \langle U_a^{(m)}, U_b^{(m)} \rangle = \rho_m^2 \delta_{ab} + O(\rho_m^{2-4\varepsilon}).$$

Proof. Let us consider the surface Σ_m equipped with the rescaled metric $\frac{1}{2\rho_m} g$. By Corollary 2.12, the intrinsic Gaussian curvature of the surface $(\Sigma_m, \frac{1}{2\rho_m} g)$ is given by $1 + O(\rho_m^{-\frac{1}{4}})$. Moreover, we have uniform bounds for the derivatives of the Gaussian curvature of $(\Sigma_m, \frac{1}{2\rho_m} g)$. Consequently, we can find a Riemannian metric γ_m on Σ_m such that (Σ_m, γ_m) has constant Gaussian curvature 1 and

$$\left\| \gamma_m - \frac{1}{2\rho_m} g|_{\Sigma_m} \right\|_{C^l(\Sigma_m, \gamma_m)} \leq C(l) \rho_m^{-4\varepsilon}.$$

Since the surface (Σ_m, γ_m) is isometric to the standard two-sphere, we can find vector fields $U_1^{(m)}, U_2^{(m)}, U_3^{(m)}$ on Σ_m such that

$$\mathcal{L}_{U_a^{(m)}}(\gamma_m) = 0$$

and

$$\int_{\Sigma_m} \langle U_a^{(m)}, U_b^{(m)} \rangle_{\gamma_m} d\text{vol}_{\gamma_m} = \frac{1}{4} \delta_{ab}.$$

It is easy to see that the vector fields $U_1^{(m)}, U_2^{(m)}, U_3^{(m)}$ have all the required properties.

In the next step, we extend the vector fields $U_1^{(m)}, U_2^{(m)}, U_3^{(m)}$ to the region Ω_m by requiring that

$$[U_a^{(m)}, X] = 0$$

for each $a \in \{1, 2, 3\}$.

Proposition 3.2. *The vector fields $U_1^{(m)}, U_2^{(m)}, U_3^{(m)}$ satisfy the following properties:*

- (i) $\sup_{\Omega_m} |D^l U_a^{(m)}| \leq C(l) \rho_m^{\frac{1-l}{2}}.$
- (ii) $\sup_{\Omega_m} |\mathcal{L}_{U_a^{(m)}}(g)| \leq O(\rho_m^{-4\epsilon}).$
- (iii) $\sup_{\Omega_m} |\Delta U_a^{(m)} + D_X U_a^{(m)}| \leq O(\rho_m^{-\frac{1}{2}-2\epsilon}).$
- (iv) $\sup_{\Omega_m} |\langle U_a^{(m)}, X \rangle| \leq O(1).$
- (v) *We have*

$$\int_{\{f=r\}} \langle U_a^{(m)}, U_b^{(m)} \rangle = r^2 \delta_{ab} + O(\rho_m^{2-4\epsilon})$$

for all $r \in [\rho_m, 4\rho_m]$.

Proof. (i) Let $g(t) = \Phi_t^*(g)$. It follows from Shi's estimates that

$$\sup_{t \in [-4\rho_m, 0]} \sup_{\{\rho_m - \sqrt{\rho_m} \leq f \leq \rho_m + \sqrt{\rho_m}\}} |D_{g(t)}^l \text{Ric}_{g(t)}|_{g(t)} \leq C(l) \rho_m^{-\frac{l+2}{2}},$$

where $D_{g(t)}$ denotes the Levi-Civita connection with respect to the metric $g(t)$. Using results from Appendix A of [2], we obtain uniform bounds for the covariant derivatives of $g(t)$ with respect to the fixed metric g . More precisely, we have

$$\sup_{t \in [-4\rho_m, 0]} \sup_{\{\rho_m - \sqrt{\rho_m} \leq f \leq \rho_m + \sqrt{\rho_m}\}} |D^l g(t)| \leq C(l) \rho_m^{-\frac{l}{2}},$$

where D denotes the Levi-Civita connection of (M, g) . Moreover, we have

$$\sup_{\Sigma_m} |D^l U_a^{(m)}| \leq C(l) \rho_m^{\frac{1-l}{2}}$$

by definition of the vector field $U_a^{(m)}$. Putting these facts together, we obtain

$$\sup_{t \in [-4\rho_m, 0]} \sup_{\Sigma_m} |D_{g(t)}^l U_a^{(m)}| \leq C(l) \rho_m^{\frac{1-l}{2}}.$$

Since $\Phi_t^*(U_a^{(m)}) = U_a^{(m)}$, we conclude that

$$\sup_{t \in [-4\rho_m, 0]} \sup_{\Phi_t(\Sigma_m)} |D^l U_a^{(m)}| \leq C(l) \rho_m^{\frac{1-l}{2}}.$$

From this, the assertion follows easily.

(ii) Let $\{e_1, e_2\}$ be a local orthonormal frame on Σ_m . By definition of $U_a^{(m)}$, we have

$$\begin{aligned} \langle D_{e_i} U_a^{(m)}, e_j \rangle + \langle D_{e_j} U_a^{(m)}, e_i \rangle &= O(\rho_m^{-4\epsilon}), \\ \langle D_X U_a^{(m)}, e_j \rangle + \langle D_{e_j} U_a^{(m)}, X \rangle &= \langle D_{U_a^{(m)}} X, e_j \rangle - \langle U_a^{(m)}, D_{e_j} X \rangle = 0, \\ \langle D_X U_a^{(m)}, X \rangle &= \frac{1}{2} \mathcal{L}_{U_a^{(m)}}(\langle X, X \rangle) = -\frac{1}{2} \langle U_a^{(m)}, \nabla R \rangle = O(\rho_m^{-1}) \end{aligned}$$

at each point on Σ_m . Thus, we conclude that

$$|\mathcal{L}_{U_a^{(m)}}(g)| \leq O(\rho_m^{-4\epsilon})$$

along Σ_m . Using the identity $[U_a^{(m)}, X] = 0$, we obtain

$$\mathcal{L}_X(\mathcal{L}_{U_a^{(m)}}(g)) = \mathcal{L}_{U_a^{(m)}}(\mathcal{L}_X(g)) = 2\mathcal{L}_{U_a^{(m)}}(\text{Ric})$$

in Ω_m . As in Lemma 2.6, we put $T = 2\text{Ric} - Rg + Rdf \otimes df$. This gives

$$\begin{aligned} & \mathcal{L}_X(\mathcal{L}_{U_a^{(m)}}(g)) \\ &= \mathcal{L}_{U_a^{(m)}}(R(g - df \otimes df)) + \mathcal{L}_{U_a^{(m)}}(T) \\ &= R\mathcal{L}_{U_a^{(m)}}(g - df \otimes df) + \langle U_a^{(m)}, \nabla R \rangle (g - df \otimes df) + \mathcal{L}_{U_a^{(m)}}(T) \end{aligned}$$

in Ω_m . Since $[U_a^{(m)}, X] = 0$, we have $\mathcal{L}_{U_a^{(m)}}(df) = (\mathcal{L}_{U_a^{(m)}}(g))(X, \cdot)$. In particular,

$$|\mathcal{L}_{U_a^{(m)}}(g - df \otimes df)| \leq C |\mathcal{L}_{U_a^{(m)}}(g)|.$$

By Proposition 2.11, we have

$$|\langle U_a^{(m)}, \nabla R \rangle| \leq O(\rho_m^{-\frac{5}{4}})$$

in Ω_m . Moreover, Lemma 2.6 implies that

$$|\mathcal{L}_{U_a^{(m)}}(T)| \leq |U_a^{(m)}| |DT| + 2|T| |DU_a^{(m)}| \leq O(\rho_m^{-\frac{3}{2}})$$

in Ω_m . Putting these facts together, we obtain

$$|\mathcal{L}_X(\mathcal{L}_{U_a^{(m)}}(g))| \leq C \rho_m^{-1} |\mathcal{L}_{U_a^{(m)}}(g)| + O(\rho_m^{-\frac{5}{4}}),$$

hence

$$|D_X(\mathcal{L}_{U_a^{(m)}}(g))| \leq C \rho_m^{-1} |\mathcal{L}_{U_a^{(m)}}(g)| + O(\rho_m^{-\frac{5}{4}})$$

in Ω_m . Integrating this inequality along the integral curves of X gives

$$\sup_{\Omega_m} |\mathcal{L}_{U_a^{(m)}}(g)| \leq O(\rho_m^{-4\epsilon}).$$

(iii) Consider the tensor $h_a^{(m)} = \mathcal{L}_{U_a^{(m)}}(g)$. It follows from (i) and (ii) that

$$\sup_{\Omega_m} |h_a^{(m)}| \leq O(\rho_m^{-4\epsilon})$$

and

$$\sup_{\Omega_m} |D^l h_a^{(m)}| \leq C(l) \rho_m^{-\frac{l}{2}}.$$

Using standard interpolation inequalities, we conclude that

$$\sup_{\Omega_m} |Dh_a^{(m)}| \leq O(\rho_m^{-\frac{1}{2}-2\epsilon}).$$

On the other hand, since $[U_a^{(m)}, X] = 0$, we obtain

$$\text{div}(h_a^{(m)}) - \frac{1}{2} \nabla(\text{tr } h_a^{(m)}) = \Delta U_a^{(m)} + \text{Ric}(U_a^{(m)}) = \Delta U_a^{(m)} + D_X U_a^{(m)}.$$

Putting these facts together, the assertion follows.

(iv) Note that $\langle U_a^{(m)}, X \rangle = 0$ along Σ_m . Using the identity $[U_a^{(m)}, X] = 0$, we obtain

$$\mathcal{L}_X(\langle U_a^{(m)}, X \rangle) = 2 \operatorname{Ric}(U_a^{(m)}, X) = -\langle U_a^{(m)}, \nabla R \rangle = O(\rho_m^{-1})$$

in Ω_m . Integrating this relation along the integral curves of X gives

$$|\langle U_a^{(m)}, X \rangle| \leq O(1)$$

in Ω_m .

(v) By Lemma 2.6, the tensor $T = 2 \operatorname{Ric} - Rg + Rdf \otimes df$ satisfies $|T| \leq O(r^{-\frac{3}{2}})$. Using the identity $[U_a^{(m)}, X] = [U_b^{(m)}, X] = 0$, we obtain

$$\begin{aligned} \mathcal{L}_X(\langle U_a^{(m)}, U_b^{(m)} \rangle) &= 2 \operatorname{Ric}(U_a^{(m)}, U_b^{(m)}) \\ &= R(\langle U_a^{(m)}, U_b^{(m)} \rangle - \langle U_a^{(m)}, X \rangle \langle U_b^{(m)}, X \rangle) + T(U_a^{(m)}, U_b^{(m)}) \\ &= R \langle U_a^{(m)}, U_b^{(m)} \rangle + O(\rho_m^{-\frac{1}{2}}) \end{aligned}$$

in Ω_m . Using Proposition 2.11, we obtain

$$\mathcal{L}_X(\langle U_a^{(m)}, U_b^{(m)} \rangle) = \frac{1}{f} \langle U_a^{(m)}, U_b^{(m)} \rangle + O(\rho_m^{-\frac{1}{4}})$$

in Ω_m . By Corollary 2.12, the mean curvature of the level surface $\{f = r\}$ equals $\frac{1}{r} + O(r^{-\frac{5}{4}})$. Thus, we conclude that

$$\begin{aligned} \frac{d}{dr} \left(\int_{\{f=r\}} \langle U_a^{(m)}, U_b^{(m)} \rangle \right) &= \int_{\{f=r\}} \frac{1}{|X|^2} \mathcal{L}_X(\langle U_a^{(m)}, U_b^{(m)} \rangle) + \int_{\{f=r\}} H \langle U_a^{(m)}, U_b^{(m)} \rangle \\ &= \frac{2}{r} \left(\int_{\{f=r\}} \langle U_a^{(m)}, U_b^{(m)} \rangle \right) + O(\rho_m^{\frac{3}{4}}) \end{aligned}$$

for all $r \in [\rho_m, 4\rho_m]$. Since

$$\int_{\{f=\rho_m\}} \langle U_a^{(m)}, U_b^{(m)} \rangle = \rho_m^2 \delta_{ab} + O(\rho_m^{2-4\epsilon}),$$

it follows that

$$\int_{\{f=r\}} \langle U_a^{(m)}, U_b^{(m)} \rangle = r^2 \delta_{ab} + O(\rho_m^{2-4\epsilon})$$

for all $r \in [\rho_m, 4\rho_m]$.

Proposition 3.3. *We can find vector fields U_1, U_2, U_3 on (M, g) such that $|\mathcal{L}_{U_a}(g)| \leq O(r^{-2\varepsilon})$ and $|\Delta U_a + D_X U_a| \leq O(r^{-\frac{1}{2}-2\varepsilon})$. Moreover, we have $|U_a| \leq O(r^{\frac{1}{2}})$ and*

$$\int_{\{f=r\}} \langle U_a, U_b \rangle = r^2 \delta_{ab} + O(r^{2-2\varepsilon})$$

if r is sufficiently large.

Proof. We define

$$\tilde{U}_a^{(m)} = \sum_{b=1}^3 \Lambda_{ab}^{(m)} U_b^{(m)},$$

where $\Lambda^{(m)}$ is a sequence of orthogonal 3×3 matrices. Proceeding inductively, we can choose the matrices $\Lambda^{(m)}$ in such a way that

$$(7) \quad \sup_{\{f=\rho_m\}} |\tilde{U}_a^{(m-1)} - \tilde{U}_a^{(m)}| \leq O(\rho_m^{\frac{1}{2}-2\varepsilon}).$$

Indeed, having constructed $\Lambda^{(m-1)}$, Proposition 3.2 guarantees the existence of an orthogonal 3×3 matrix $\Lambda^{(m)}$ such that (7) holds. Using the identity $[\tilde{U}_a^{(m-1)}, X] = [\tilde{U}_a^{(m)}, X] = 0$, we obtain

$$\begin{aligned} \mathcal{L}_X(|\tilde{U}_a^{(m-1)} - \tilde{U}_a^{(m)}|^2) &= 2 \operatorname{Ric}(\tilde{U}_a^{(m-1)} - \tilde{U}_a^{(m)}, \tilde{U}_a^{(m-1)} - \tilde{U}_a^{(m)}) \\ &\leq C \rho_m^{-1} |\tilde{U}_a^{(m-1)} - \tilde{U}_a^{(m)}|^2 \end{aligned}$$

in the region $\{\rho_m \leq f \leq 2\rho_m\}$. Integrating this inequality along the integral curves of X gives

$$\sup_{\{\rho_m \leq f \leq 2\rho_m\}} |\tilde{U}_a^{(m-1)} - \tilde{U}_a^{(m)}| \leq O(\rho_m^{\frac{1}{2}-2\varepsilon}).$$

We now fix a smooth cutoff function $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\zeta(s) = 1$ for $s \leq \frac{4}{3}$ and $\zeta(s) = 0$ for $s \geq \frac{5}{3}$. We define vector fields U_1, U_2, U_3 by

$$U_a = \tilde{U}_a^{(m)} + \zeta(\rho_m^{-1} f) (\tilde{U}_a^{(m-1)} - \tilde{U}_a^{(m)})$$

in the region $\{\rho_m \leq f \leq 2\rho_m\}$. It follows from Proposition 3.2 that the vector fields U_1, U_2, U_3 have the required properties.

We note that we have only defined the vector fields U_1, U_2, U_3 outside of a bounded region. Since we are only interested in the asymptotic behavior near infinity, we can extend the vector fields U_1, U_2, U_3 in an arbitrary way into the interior.

4. A PDE FOR THE LIE DERIVATIVE OF A VECTOR FIELD

In this section, we consider a vector field W satisfying $\Delta W + D_X W = 0$. Our goal is to derive an elliptic equation for the Lie derivative $\mathcal{L}_W(g)$.

Theorem 4.1. *Suppose that W is a vector field satisfying $\Delta W + D_X W = 0$. Then the Lie derivative $\mathcal{L}_W(g)$ satisfies*

$$\Delta_L(\mathcal{L}_W(g)) + \mathcal{L}_X(\mathcal{L}_W(g)) = 0.$$

Proof. Let $g(t)$ be a smooth one-parameter family of metrics with $g(0) = g$. It follows from Proposition 2.3.7 in [24] that

$$(8) \quad -2 \frac{\partial}{\partial t} \text{Ric}_{g(t)} \Big|_{t=0} = \Delta_L h - \mathcal{L}_Z(g),$$

where $h = \frac{\partial}{\partial t} g(t) \Big|_{t=0}$ and

$$Z = \text{div } h - \frac{1}{2} \nabla(\text{tr } h).$$

Let us apply the formula (8) to the family of metrics obtained by pulling back g under the one-parameter group of diffeomorphisms generated by W . This gives

$$(9) \quad -2 \mathcal{L}_W(\text{Ric}) = \Delta_L h - \mathcal{L}_Z(g),$$

where $h = \mathcal{L}_W(g)$ and

$$Z = \text{div } h - \frac{1}{2} \nabla(\text{tr } h) = \Delta W + \text{Ric}(W).$$

Using the relation $\Delta W + D_X W = 0$, we obtain

$$Z = \Delta W + D_W X = -[X, W].$$

Substituting this identity into (9), we conclude that

$$\begin{aligned} \Delta_L(\mathcal{L}_W(g)) &= -2 \mathcal{L}_W(\text{Ric}) + \mathcal{L}_Z(g) \\ &= -\mathcal{L}_W(\mathcal{L}_X(g)) - \mathcal{L}_{[X, W]}(g) \\ &= -\mathcal{L}_X(\mathcal{L}_W(g)). \end{aligned}$$

This completes the proof.

Applying Theorem 4.1 to the vector field X gives the following result:

Proposition 4.2. *The vector field X satisfies $\Delta X + D_X X = 0$. Moreover, the Ricci tensor satisfies*

$$\Delta_L(\text{Ric}) + \mathcal{L}_X(\text{Ric}) = 0.$$

Proof. Let $h = \mathcal{L}_X(g) = 2 \text{Ric}$. The contracted second Bianchi identity implies that

$$0 = \text{div } h - \frac{1}{2} \nabla(\text{tr } h) = \Delta X + \text{Ric}(X) = \Delta X + D_X X.$$

Using Theorem 4.1, we obtain

$$\Delta_L h + \mathcal{L}_X(h) = 0,$$

as claimed.

We note that the identity $\Delta_L(\text{Ric}) + \mathcal{L}_X(\text{Ric}) = 0$ can alternatively be derived from the evolution equation for the Ricci tensor under the Ricci flow (see e.g. [2], Chapter 2).

5. AN ELLIPTIC PDE FOR VECTOR FIELDS

Throughout this section, we fix a smooth vector field Q on M such that $|Q| \leq O(r^{-\frac{1}{2}-2\varepsilon})$. Our goal is to construct a vector field V on M such that $\Delta V + D_X V = Q$ and $|V| \leq O(r^{\frac{1}{2}-\varepsilon})$.

Proposition 5.1. *Let V be a smooth vector field satisfying $\Delta V + D_X V = Q$ in the region $\{f \leq \rho\}$. Then*

$$\sup_{\{f \leq \rho\}} |V| \leq \sup_{\{f = \rho\}} |V| + B \rho^{\frac{1}{2}-2\varepsilon}$$

for some uniform constant $B \geq 1$.

Proof. It follows from Kato's inequality that

$$\begin{aligned} \Delta(|V|^2) + \langle X, \nabla(|V|^2) \rangle &= 2|DV|^2 + 2\langle V, Q \rangle \\ &\geq 2|\nabla|V||^2 - 2|Q||V|. \end{aligned}$$

This implies

$$\Delta|V| + \langle X, \nabla|V| \rangle \geq -|Q|$$

when $V \neq 0$. Moreover, using the identity $\Delta f + |\nabla f|^2 = 1$ and the inequality $f \geq 1$, we obtain

$$\begin{aligned} &\Delta(f^{\frac{1}{2}-2\varepsilon}) + \langle X, \nabla(f^{\frac{1}{2}-2\varepsilon}) \rangle \\ &= \left(\frac{1}{2} - 2\varepsilon\right) f^{-\frac{1}{2}-2\varepsilon} - \left(\frac{1}{4} - 4\varepsilon^2\right) f^{-\frac{3}{2}-2\varepsilon} |\nabla f|^2 \\ &\geq \left(\frac{1}{2} - 2\varepsilon\right) f^{-\frac{1}{2}-2\varepsilon} - \left(\frac{1}{4} - 4\varepsilon^2\right) f^{-\frac{1}{2}-2\varepsilon} \\ &= \left(\frac{1}{2} - 2\varepsilon\right)^2 f^{-\frac{1}{2}-2\varepsilon}. \end{aligned}$$

By assumption, we can find a constant $B \geq 1$ such that

$$|Q| < \left(\frac{1}{2} - 2\varepsilon\right)^2 B f^{-\frac{1}{2}-2\varepsilon}.$$

Putting these facts together, we obtain

$$\Delta(|V| + B f^{\frac{1}{2}-2\varepsilon}) + \langle X, \nabla(|V| + B f^{\frac{1}{2}-2\varepsilon}) \rangle > 0$$

when $V \neq 0$. Using the maximum principle, we conclude that

$$\sup_{\{f \leq \rho\}} (|V| + B f^{\frac{1}{2}-2\varepsilon}) \leq \sup_{\{f = \rho\}} |V| + B \rho^{\frac{1}{2}-2\varepsilon}.$$

From this, the assertion follows.

We now consider a sequence of real numbers $\rho_m \rightarrow \infty$. Given any integer m , there exists a unique vector field $V^{(m)}$ such that $\Delta V^{(m)} + D_X V^{(m)} = Q$

in the region $\{f \leq \rho_m\}$ and $V^{(m)} = 0$ on the boundary $\{f = \rho_m\}$. Moreover, we define

$$A^{(m)}(r) = \inf_{\lambda \in \mathbb{R}} \sup_{\{f=r\}} |V^{(m)} - \lambda X|$$

for $r \leq \rho_m$.

Proposition 5.2. *Let us fix $\tau \in (0, \frac{1}{2})$ sufficiently small. Then we can find a real number ρ_0 and a positive integer m_0 such that*

$$2\tau^{-\frac{1}{2}+\varepsilon} A^{(m)}(\tau r) \leq A^{(m)}(r) + r^{\frac{1}{2}-\varepsilon}$$

for all $r \in [\rho_0, \rho_m]$ and all $m \geq m_0$.

Proof. Suppose that the assertion is false. After passing to a subsequence if necessary, we can find a sequence of real numbers $r_m \leq \rho_m$ such that $r_m \rightarrow \infty$ and

$$A^{(m)}(r_m) + r_m^{\frac{1}{2}-\varepsilon} \leq 2\tau^{-\frac{1}{2}+\varepsilon} A^{(m)}(\tau r_m)$$

for all m . For each m , we choose a real number λ_m such that

$$\sup_{\{f=r_m\}} |V^{(m)} - \lambda_m X| = A^{(m)}(r_m).$$

The vector field $V^{(m)} - \lambda_m X$ satisfies the equation

$$\Delta(V^{(m)} - \lambda_m X) + D_X(V^{(m)} - \lambda_m X) = Q.$$

Using Proposition 5.1, we obtain

$$\begin{aligned} \sup_{\{f \leq r_m\}} |V^{(m)} - \lambda_m X| &\leq \sup_{\{f=r_m\}} |V^{(m)} - \lambda_m X| + B r_m^{\frac{1}{2}-2\varepsilon} \\ &= A^{(m)}(r_m) + B r_m^{\frac{1}{2}-2\varepsilon}. \end{aligned}$$

Therefore, the vector field

$$\tilde{V}^{(m)} = \frac{1}{A^{(m)}(r_m) + r_m^{\frac{1}{2}-\varepsilon}} (V^{(m)} - \lambda_m X)$$

satisfies

$$(10) \quad \sup_{\{f \leq r_m\}} |\tilde{V}^{(m)}| \leq \frac{A^{(m)}(r_m) + B r_m^{\frac{1}{2}-2\varepsilon}}{A^{(m)}(r_m) + r_m^{\frac{1}{2}-\varepsilon}} \leq B$$

and

$$\begin{aligned} (11) \quad \inf_{\lambda \in \mathbb{R}} \sup_{\{f=\tau r_m\}} |\tilde{V}^{(m)} - \lambda X| &= \frac{1}{A^{(m)}(r_m) + r_m^{\frac{1}{2}-\varepsilon}} \inf_{\lambda \in \mathbb{R}} \sup_{\{f=\tau r_m\}} |V^{(m)} - \lambda X| \\ &= \frac{A^{(m)}(\tau r_m)}{A^{(m)}(r_m) + r_m^{\frac{1}{2}-\varepsilon}} \\ &\geq \frac{1}{2} \tau^{\frac{1}{2}-\varepsilon}. \end{aligned}$$

We next define

$$\hat{g}^{(m)}(t) = r_m^{-1} \Phi_{r_m t}^*(g)$$

and

$$\hat{V}^{(m)}(t) = r_m^{\frac{1}{2}} \Phi_{r_m t}^*(\tilde{V}^{(m)}).$$

Since (M, g) is a steady Ricci soliton, the metrics $\hat{g}^{(m)}(t)$ form a solution to the Ricci flow. Moreover, the vector fields $\hat{V}^{(m)}(t)$ satisfy the parabolic equation

$$\frac{\partial}{\partial t} \hat{V}^{(m)}(t) = \Delta_{\hat{g}^{(m)}(t)} \hat{V}^{(m)}(t) + \text{Ric}_{\hat{g}^{(m)}(t)}(\hat{V}^{(m)}(t)) - \hat{Q}^{(m)}(t),$$

where

$$\hat{Q}^{(m)}(t) = \frac{r_m^{\frac{3}{2}}}{A^{(m)}(r_m) + r_m^{\frac{1}{2}-\varepsilon}} \Phi_{r_m t}^*(Q).$$

The inequality (10) implies that

$$\limsup_{m \rightarrow \infty} \sup_{t \in [\delta, 1-\delta]} \sup_{\{r_m - \delta^{-1} \sqrt{r_m} \leq f \leq r_m + \delta^{-1} \sqrt{r_m}\}} |\hat{V}^{(m)}(t)|_{\hat{g}^{(m)}(t)} < \infty$$

for any given $\delta \in (0, \frac{1}{2})$. Moreover, using the estimate $|Q| \leq O(r^{-\frac{1}{2}-2\varepsilon})$, we obtain

$$\limsup_{m \rightarrow \infty} \sup_{t \in [\delta, 1-\delta]} \sup_{\{r_m - \delta^{-1} \sqrt{r_m} \leq f \leq r_m + \delta^{-1} \sqrt{r_m}\}} |\hat{Q}^{(m)}(t)|_{\hat{g}^{(m)}(t)} = 0$$

for any given $\delta \in (0, \frac{1}{2})$.

We now pass to the limit as $m \rightarrow \infty$. To that end, we choose a sequence of marked points $p_m \in M$ such that $f(p_m) = r_m$. The sequence $(M, \hat{g}^{(m)}(t), p_m)$ converges in the Cheeger-Gromov sense to a one-parameter family of shrinking cylinders $(S^2 \times \mathbb{R}, \bar{g}(t))$, $t \in (0, 1)$, where $\bar{g}(t)$ is given by (2). The rescaled vector fields $r_m^{\frac{1}{2}} X$ converge to the axial vector field $\frac{\partial}{\partial z}$ on $S^2 \times \mathbb{R}$. Finally, the vector fields $\hat{V}^{(m)}(t)$ converge to a one-parameter family of vector fields $\bar{V}(t)$, $t \in (0, 1)$, which satisfy the parabolic equation

$$(12) \quad \frac{\partial}{\partial t} \bar{V}(t) = \Delta_{\bar{g}(t)} \bar{V}(t) + \text{Ric}_{\bar{g}(t)}(\bar{V}(t)).$$

Using the identity

$$\Phi_{\sqrt{r_m} s}^*(\hat{V}^{(m)}(t)) = \hat{V}^{(m)}\left(t + \frac{s}{\sqrt{r_m}}\right),$$

we conclude that $\Psi_s^*(\bar{V}(t)) = \bar{V}(t)$, where $\Psi_s : S^2 \times \mathbb{R} \rightarrow S^2 \times \mathbb{R}$ denotes the flow generated by the axial vector field $-\frac{\partial}{\partial z}$. Hence, $\bar{V}(t)$ is invariant under translations along the axis of the cylinder. Using the estimate (10), we obtain

$$(13) \quad |\bar{V}(t)|_{\bar{g}(t)} \leq B,$$

where B is a positive constant that does not depend on τ . Finally, the inequality (11) implies

$$\inf_{\lambda \in \mathbb{R}} \sup_{\Phi_{r_m(\tau-1)}(\{f=\tau r_m\})} \left| \hat{V}^{(m)}(1-\tau) - \lambda r_m^{\frac{1}{2}} X \right|_{\hat{g}^{(m)}(1-\tau)} \geq \frac{1}{2} \tau^{\frac{1}{2}-\varepsilon},$$

hence

$$(14) \quad \inf_{\lambda \in \mathbb{R}} \sup_{S^2 \times \mathbb{R}} \left| \overline{V}(1-\tau) - \lambda \frac{\partial}{\partial z} \right|_{\overline{g}(1-\tau)} \geq \frac{1}{2} \tau^{\frac{1}{2}-\varepsilon}.$$

We will show that this situation leads to a contradiction, provided that we choose τ sufficiently small. Since $\overline{V}(t)$ is invariant under translations along the axis of the cylinder, we may write

$$\overline{V}(t) = \xi(t) + \eta(t) \frac{\partial}{\partial z}$$

for $t \in (0, 1)$, where $\xi(t)$ is a vector field on S^2 and $\eta(t)$ is a real-valued function on S^2 . The parabolic equation (12) is equivalent to the following system of equations ξ and η :

$$(15) \quad \frac{\partial}{\partial t} \xi(t) = \frac{1}{2-2t} (\Delta_{S^2} \xi(t) + \xi(t)),$$

$$(16) \quad \frac{\partial}{\partial t} \eta(t) = \frac{1}{2-2t} \Delta_{S^2} \eta(t),$$

Finally, the estimate (13) implies

$$(17) \quad \sup_{S^2} |\xi(t)|_{g_{S^2}} \leq L_1 (1-t)^{-\frac{1}{2}},$$

$$(18) \quad \sup_{S^2} |\eta(t)| \leq L_1$$

for each $t \in (0, 1)$, where L_1 is a uniform constant that does not depend on τ .

Consider now the Laplacian on S^2 , acting on vector fields. It follows from Proposition A.1 that the first eigenvalue of this operator is at least 1. Using (15) and (17), we conclude that

$$(19) \quad \sup_{S^2} |\xi(t)|_{g_{S^2}} \leq L_2$$

for all $t \in [\frac{1}{2}, 1)$, where L_2 is a uniform constant that does not depend on τ . Similarly, using (16) and (18), we can show that

$$(20) \quad \inf_{\lambda \in \mathbb{R}} \sup_{S^2} |\eta(t) - \lambda|_{g_{S^2}} \leq L_3 (1-t)$$

for each $t \in [\frac{1}{2}, 1)$, where L_3 is a uniform constant that does not depend on τ . Combining (19) and (20), we conclude that

$$(21) \quad \inf_{\lambda \in \mathbb{R}} \sup_{S^2 \times \mathbb{R}} \left| \overline{V}(t) - \lambda \frac{\partial}{\partial z} \right|_{\overline{g}(t)} \leq L_4 (1-t)^{\frac{1}{2}}$$

for each $t \in [\frac{1}{2}, 1)$, where L_4 is a uniform constant that does not depend on τ .

Hence, if we choose $\tau > 0$ sufficiently small so that $\tau^{-\varepsilon} > 2L_4$, then the inequalities (14) and (21) are in contradiction. This completes the proof of Proposition 5.2.

Proposition 5.3. *There exists a sequence of real numbers λ_m such that*

$$\sup_m \sup_{\{f \leq \rho_m\}} f^{-\frac{1}{2}+\varepsilon} |V^{(m)} - \lambda_m X| < \infty.$$

Proof. Let us fix $\tau \in (0, \frac{1}{2})$ sufficiently small so that the conclusion of Proposition 5.2 holds. Then we can find a real number ρ_0 and a positive integer m_0 such that

$$2\tau^{-\frac{1}{2}+\varepsilon} A^{(m)}(\tau r) \leq A^{(m)}(r) + r^{\frac{1}{2}-\varepsilon}$$

for all $r \in [\rho_0, \rho_m]$ and all $m \geq m_0$. Moreover, Proposition 5.1 implies that

$$\sup_{\rho_0 \leq r \leq \rho_m} A^{(m)}(r) \leq \sup_{\{f \leq \rho_m\}} |V| \leq B \rho_m^{\frac{1}{2}-2\varepsilon}.$$

Putting these facts together, we conclude that

$$(22) \quad \sup_{m \geq m_0} \sup_{\rho_0 \leq r \leq \rho_m} r^{-\frac{1}{2}+\varepsilon} A^{(m)}(r) < \infty.$$

We may assume that ρ_0 is chosen large enough so that $\sup_{\{f=\rho_0\}} |X| \geq \frac{1}{2}$. We next choose a sequence of real numbers λ_m such that

$$\sup_{\{f=\rho_0\}} |V^{(m)} - \lambda_m X| = A^{(m)}(\rho_0).$$

Applying Proposition 5.1 to the vector field $V^{(m)} - \lambda X$, we obtain

$$\sup_{\{f=\rho_0\}} |V^{(m)} - \lambda X| \leq \sup_{\{f=r\}} |V^{(m)} - \lambda X| + B r^{\frac{1}{2}-2\varepsilon}$$

for all $r \in [\rho_0, \rho_m]$ and all $\lambda \in \mathbb{R}$. This implies

$$\begin{aligned} & \sup_{\{f=r\}} |V^{(m)} - \lambda_m X| \\ & \leq \sup_{\{f=r\}} |V^{(m)} - \lambda X| + |\lambda - \lambda_m| \\ & \leq \sup_{\{f=r\}} |V^{(m)} - \lambda X| + 2 \sup_{\{f=\rho_0\}} |\lambda X - \lambda_m X| \\ & \leq \sup_{\{f=r\}} |V^{(m)} - \lambda X| + 2 \sup_{\{f=\rho_0\}} |V^{(m)} - \lambda_m X| + 2 \sup_{\{f=\rho_0\}} |V^{(m)} - \lambda X| \\ & \leq 3 \sup_{\{f=r\}} |V^{(m)} - \lambda X| + 2 \sup_{\{f=\rho_0\}} |V^{(m)} - \lambda_m X| + 2B r^{\frac{1}{2}-2\varepsilon} \end{aligned}$$

for all $r \in [\rho_0, \rho_m]$ and all $\lambda \in \mathbb{R}$. Taking the infimum over $\lambda \in \mathbb{R}$ gives

$$\sup_{\{f=r\}} |V^{(m)} - \lambda_m X| \leq 3A^{(m)}(r) + 2A^{(m)}(\rho_0) + 2Br^{\frac{1}{2}-2\varepsilon}$$

for all $r \in [\rho_0, \rho_m]$. Consequently, the inequality (22) implies

$$\sup_{m \geq m_0} \sup_{\rho_0 \leq r \leq \rho_m} \sup_{\{f=r\}} r^{-\frac{1}{2}+\varepsilon} |V^{(m)} - \lambda_m X| < \infty.$$

From this, the assertion follows easily.

Theorem 5.4. *There exists a some vector field V such that $\Delta V + D_X V = Q$ and $|V| \leq O(r^{\frac{1}{2}-\varepsilon})$. Moreover, $|DV| \leq O(r^{-\varepsilon})$.*

Proof. By Proposition 5.3, we can find a sequence of real numbers λ_m such that

$$\sup_m \sup_{\{f \leq \rho_m\}} f^{-\frac{1}{2}+\varepsilon} |V^{(m)} - \lambda_m X| < \infty.$$

Moreover, the vector field $V^{(m)} - \lambda_m X$ solves the equation

$$\Delta(V^{(m)} - \lambda_m X) + D_X(V^{(m)} - \lambda_m X) = Q$$

in the region $\{f \leq \rho_m\}$. Hence, after passing to a subsequence if necessary, the vector fields $V^{(m)} - \lambda_m X$ converge to a smooth vector field V satisfying $\Delta V + D_X V = Q$ and $|V| \leq O(r^{\frac{1}{2}-\varepsilon})$.

It remains to show that $|DV| \leq O(r^{-\varepsilon})$. In order to prove this, we use the standard interior regularity theory for parabolic equations. Consider a sequence $r_m \rightarrow \infty$, and let

$$\hat{g}^{(m)}(t) = r_m^{-1} \Phi_{r_m t}^*(g)$$

for $t \in [-\frac{1}{2}, 0]$. Moreover, we define

$$\hat{V}^{(m)}(t) = \Phi_{r_m t}^*(V)$$

and

$$\hat{Q}^{(m)}(t) = r_m \Phi_{r_m t}^*(Q)$$

for $t \in [-\frac{1}{2}, 0]$. The vector fields $\hat{V}^{(m)}(t)$ satisfy the parabolic equation

$$\frac{\partial}{\partial t} \hat{V}^{(m)}(t) = \Delta_{\hat{g}^{(m)}(t)} \hat{V}^{(m)}(t) + \text{Ric}_{\hat{g}^{(m)}(t)}(\hat{V}^{(m)}(t)) - \hat{Q}^{(m)}(t).$$

Moreover, since $|Q| \leq O(r^{-\frac{1}{2}-2\varepsilon})$, we have

$$\sup_{t \in [-\frac{1}{2}, 0]} \sup_{\{r_m - \sqrt{r_m} \leq f \leq r_m + \sqrt{r_m}\}} |\hat{Q}^{(m)}(t)|_{\hat{g}^{(m)}(t)} \leq O(r_m^{-2\varepsilon}).$$

Using standard interior estimates for parabolic equations, we obtain

$$\begin{aligned} \sup_{\{f=r_m\}} |D\hat{V}^{(m)}(0)|_{\hat{g}^{(m)}(0)} &\leq C \sup_{t \in [-\frac{1}{2}, 0]} \sup_{\{r_m - \sqrt{r_m} \leq f \leq r_m + \sqrt{r_m}\}} |\hat{V}^{(m)}(t)|_{\hat{g}^{(m)}(t)} \\ &\quad + C \sup_{t \in [-\frac{1}{2}, 0]} \sup_{\{r_m - \sqrt{r_m} \leq f \leq r_m + \sqrt{r_m}\}} |\hat{Q}^{(m)}(t)|_{\hat{g}^{(m)}(t)} \\ &\leq O(r_m^{-\varepsilon}). \end{aligned}$$

From this, we deduce that

$$\sup_{\{f=r_m\}} |DV| \leq O(r_m^{-\varepsilon}),$$

as claimed.

6. ANALYSIS OF THE LICHNEROWICZ EQUATION

Proposition 6.1. *Let h be a solution of the Lichnerowicz-type equation*

$$\Delta_L h + \mathcal{L}_X(h) = 0$$

on the region $\{f \leq \rho\}$. Then

$$\sup_{\{f \leq \rho\}} \frac{|h|}{R} \leq \sup_{\{f=\rho\}} \frac{|h|}{R}.$$

Proof. By a result of Anderson and Chow [1], we have

$$\Delta\left(\frac{|h|^2}{R^2}\right) + \left\langle X + 2 \frac{\nabla R}{R}, \nabla\left(\frac{|h|^2}{R^2}\right) \right\rangle \geq 0.$$

Hence, the assertion follows from the maximum principle.

Theorem 6.2. *Let h be a solution of the Lichnerowicz-type equation*

$$\Delta_L h + \mathcal{L}_X(h) = 0.$$

such that $|h| \leq O(r^{-\varepsilon})$. Then $h = \lambda \text{Ric}$ for some constant $\lambda \in \mathbb{R}$.

Proof. Let

$$A(r) = \inf_{\lambda \in \mathbb{R}} \sup_{\{f=r\}} |h - \lambda \text{Ric}|.$$

Clearly, $A(r) \leq \sup_{\{f=r\}} |h| \leq O(r^{-\varepsilon})$. We consider two cases:

Case 1: Suppose that there exists a sequence of real numbers $r_m \rightarrow \infty$ such that $A(r_m) = 0$ for all m . For each m , we choose a real number λ_m such that

$$\sup_{\{f=r_m\}} |h - \lambda_m \text{Ric}| = A(r_m) = 0.$$

Applying Proposition 6.1 to the tensor $h - \lambda_m \text{Ric}$, we obtain

$$\sup_{\{f \leq r_m\}} \frac{|h - \lambda_m \text{Ric}|}{R} \leq \sup_{\{f=r_m\}} \frac{|h - \lambda_m \text{Ric}|}{R} = 0.$$

Therefore, we have $h - \lambda_m \text{Ric} = 0$ in the region $\{f \leq r_m\}$. Consequently, the sequence λ_m is constant and h is a constant multiple of the Ricci tensor.

Case 2: Suppose now that $A(r) > 0$ when r is sufficiently large. Let us fix a small number $\tau \in (0, \frac{1}{2})$. Since $A(r) \leq O(r^{-\varepsilon})$, we can find a sequence of real numbers $r_m \rightarrow \infty$ such that

$$A(r_m) \leq 2\tau^\varepsilon A(\tau r_m)$$

for all m . For each m , we choose a real number λ_m such that

$$\sup_{\{f=r_m\}} |h - \lambda_m \text{Ric}| = A(r_m).$$

The tensor

$$\tilde{h}^{(m)} = \frac{1}{A(r_m)} (h - \lambda_m \text{Ric})$$

satisfies the Lichnerowicz-type equation

$$\Delta_L \tilde{h}^{(m)} + \mathcal{L}_X(\tilde{h}^{(m)}) = 0.$$

Using Proposition 6.1, we obtain

$$\sup_{\{f=r\}} \frac{|\tilde{h}^{(m)}|}{R} \leq \sup_{\{f=r_m\}} \frac{|\tilde{h}^{(m)}|}{R} = \frac{1}{A(r_m)} \sup_{\{f=r_m\}} \frac{|h - \lambda_m \text{Ric}|}{R} \leq \sup_{\{f=r_m\}} \frac{1}{R}$$

for $r \leq r_m$. From this, we deduce that

$$(23) \quad \sup_{\{f=r\}} |\tilde{h}^{(m)}| \leq B \frac{r_m}{r},$$

where B is a positive constant that does not depend on τ . Finally, we have

$$\begin{aligned} \inf_{\lambda \in \mathbb{R}} \sup_{\{f=\tau r_m\}} |\tilde{h}^{(m)} - \lambda r_m \text{Ric}| &= \frac{1}{A(r_m)} \inf_{\lambda \in \mathbb{R}} \sup_{\{f=\tau r_m\}} |h^{(m)} - \lambda \text{Ric}| \\ (24) \quad &= \frac{A(\tau r_m)}{A(r_m)} \\ &\geq \frac{1}{2} \tau^{-\varepsilon}. \end{aligned}$$

We now define

$$\hat{g}^{(m)}(t) = r_m^{-1} \Phi_{r_m t}^*(g)$$

and

$$\hat{h}^{(m)}(t) = r_m^{-1} \Phi_{r_m t}^*(\tilde{h}^{(m)}).$$

Since (M, g) is a steady Ricci soliton, the metrics $\hat{g}^{(m)}(t)$ evolve by the Ricci flow. Moreover, the tensors $\hat{h}^{(m)}(t)$ satisfy the parabolic Lichnerowicz equation

$$\frac{\partial}{\partial t} \hat{h}^{(m)}(t) = \Delta_{L, \hat{g}^{(m)}(t)} \hat{h}^{(m)}(t).$$

It follows from (23) that

$$\limsup_{m \rightarrow \infty} \sup_{t \in [\delta, 1-\delta]} \sup_{\{r_m - \delta^{-1} \sqrt{r_m} \leq f \leq r_m + \delta^{-1} \sqrt{r_m}\}} |\hat{h}^{(m)}(t)|_{\hat{g}^{(m)}(t)} < \infty$$

for any given $\delta \in (0, \frac{1}{2})$.

We next take the limit as $m \rightarrow \infty$. As above, we choose a sequence of marked points $p_m \in M$ such that $f(p_m) = r_m$. The sequence $(M, \hat{g}^{(m)}(t), p_m)$ converges in the Cheeger-Gromov sense to a one-parameter family of shrinking cylinders $(S^2 \times \mathbb{R}, \bar{g}(t))$, $t \in (0, 1)$, where $\bar{g}(t)$ is given by (2). The rescaled vector fields $r_m^{\frac{1}{2}} X$ converge to the axial vector field $\frac{\partial}{\partial z}$ on $S^2 \times \mathbb{R}$. Finally, the tensors $\hat{h}^{(m)}(t)$ converge to a one-parameter family of tensors $\bar{h}(t)$, $t \in (0, 1)$, satisfying the parabolic Lichnerowicz equation

$$(25) \quad \frac{\partial}{\partial t} \bar{h}(t) = \Delta_{L, \bar{g}(t)} \bar{h}(t).$$

Using the identity

$$\Phi_{\sqrt{r_m} s}^* (\hat{h}^{(m)}(t)) = \hat{h}^{(m)} \left(t + \frac{s}{\sqrt{r_m}} \right),$$

we obtain $\Psi_s^* (\bar{h}(t)) = \bar{h}(t)$, where $\Psi_s : S^2 \times \mathbb{R} \rightarrow S^2 \times \mathbb{R}$ denotes the flow generated by the axial vector field $-\frac{\partial}{\partial z}$. In other words, $\bar{h}(t)$ is invariant under translations along the axis of the cylinder. Moreover, the estimate (23) implies

$$(26) \quad |\bar{h}(t)|_{\bar{g}(t)} \leq B (1-t)^{-1},$$

where B is a positive constant that does not depend on τ . Finally, the inequality (24) implies

$$\inf_{\lambda \in \mathbb{R}} \sup_{\Phi_{r_m(\tau-1)}(\{f=\tau r_m\})} \left| \hat{h}^{(m)}(1-\tau) - \lambda \text{Ric}_{\hat{g}^{(m)}(1-\tau)} \right|_{\hat{g}^{(m)}(1-\tau)} \geq \frac{1}{2} \tau^{-\varepsilon},$$

hence

$$(27) \quad \inf_{\lambda \in \mathbb{R}} \sup_{S^2 \times \mathbb{R}} |\bar{h}(1-\tau) - \lambda \text{Ric}_{\bar{g}(1-\tau)}|_{\bar{g}(1-\tau)} \geq \frac{1}{2} \tau^{-\varepsilon}.$$

We will show that this situation leads to a contradiction if we choose τ sufficiently small. Since $\bar{h}(t)$ is invariant under translations along the axis of the cylinder, we may write

$$\bar{h}(t) = \chi(t) + dz \otimes \sigma(t) + \sigma(t) \otimes dz + \beta(t) dz \otimes dz$$

for $t \in (0, 1)$, where $\chi(t)$ is a symmetric $(0, 2)$ tensor on S^2 , $\sigma(t)$ is a one-form on S^2 , and $\beta(t)$ is a real-valued function on S^2 . The parabolic Lichnerowicz equation (25) is equivalent to the following system of equations for χ , σ , and β :

$$(28) \quad \frac{\partial}{\partial t} \chi(t) = \frac{1}{2-2t} (\Delta_{S^2} \chi(t) - 4 \overset{\circ}{\chi}(t)),$$

$$(29) \quad \frac{\partial}{\partial t} \sigma(t) = \frac{1}{2-2t} \Delta_{S^2} \sigma(t),$$

$$(30) \quad \frac{\partial}{\partial t} \beta(t) = \frac{1}{2-2t} \Delta_{S^2} \beta(t).$$

Here, $\overset{\circ}{\chi}(t)$ denotes the trace-free part of $\chi(t)$ with respect to the standard metric on S^2 . Finally, the estimate (26) implies

$$(31) \quad \sup_{S^2} |\chi(t)|_{g_{S^2}} \leq N_1,$$

$$(32) \quad \sup_{S^2} |\sigma(t)|_{g_{S^2}} \leq N_1 (1-t)^{-\frac{1}{2}},$$

$$(33) \quad \sup_{S^2} |\beta(t)| \leq N_1 (1-t)^{-1}$$

for each $t \in (0, 1)$, where N_1 is a uniform constant that does not depend on τ .

Let us consider the operator $\chi \mapsto \Delta_{S^2} \chi - 4 \overset{\circ}{\chi}$, acting on symmetric $(0, 2)$ -tensors on S^2 . The first eigenvalue of this operator is equal to 0, and the associated eigenspace is spanned by g_{S^2} . Moreover, all other eigenvalues are at least 2 (cf. Proposition A.2 below). Hence, it follows from (28) and (31) that

$$(34) \quad \inf_{\lambda \in \mathbb{R}} \sup_{S^2} |\chi(t) - \lambda g_{S^2}|_{g_{S^2}} \leq N_2 (1-t)$$

for all $t \in [\frac{1}{2}, 1)$, where N_2 is a uniform constant that does not depend on τ . We next consider the Laplacian on S^2 , acting on one-forms. By Proposition A.1, the first eigenvalue of this operator is at least 1. Using (29) and (32), we deduce that

$$(35) \quad \sup_{S^2} |\sigma(t)|_{g_{S^2}} \leq N_3 (1-t)^{\frac{1}{2}}$$

for all $t \in [\frac{1}{2}, 1)$, where N_3 is a uniform constant that does not depend on τ . Finally, using (30) and (33), we obtain

$$(36) \quad \sup_{S^2} |\beta(t)| \leq N_4$$

for all $t \in [\frac{1}{2}, 1)$, where N_4 is a uniform constant that does not depend on τ . Combining (34), (35), and (36), we conclude that

$$(37) \quad \inf_{\lambda \in \mathbb{R}} \sup_{S^2 \times \mathbb{R}} |\bar{h}(t) - \lambda \text{Ric}_{\bar{g}(t)}|_{\bar{g}(t)} \leq N_5$$

for each $t \in [\frac{1}{2}, 1)$, where N_5 is a uniform constant that does not depend on τ .

Hence, if we choose $\tau > 0$ sufficiently small so that $\tau^{-\varepsilon} > 2N_5$, then the inequalities (27) and (37) are in contradiction. This completes the proof of Theorem 6.2.

7. PROOF OF THEOREM 1.1

Combining Theorems 4.1, 5.4, and 6.2, we obtain the following symmetry principle:

Theorem 7.1. *Suppose that U is a vector field on (M, g) such that $|\mathcal{L}_U(g)| \leq O(r^{-2\varepsilon})$ and $|\Delta U + D_X U| \leq O(r^{-\frac{1}{2}-2\varepsilon})$ for some small constant $\varepsilon > 0$. Then there exists a vector field \hat{U} on (M, g) such that $\mathcal{L}_{\hat{U}}(g) = 0$, $[\hat{U}, X] = 0$, $\langle \hat{U}, X \rangle = 0$, and $|\hat{U} - U| \leq O(r^{\frac{1}{2}-\varepsilon})$.*

Proof. By Theorem 5.4, we can find a smooth vector field V such that

$$\Delta V + D_X V = \Delta U + D_X U$$

and $|V| \leq O(r^{\frac{1}{2}-\varepsilon})$. Moreover, the covariant derivative of V satisfies $|DV| \leq O(r^{-\varepsilon})$. We now define $W = U - V$ and $h = \mathcal{L}_W(g)$. Since W satisfies the equation $\Delta W + D_X W = 0$, Theorem 4.1 implies that the tensor h satisfies the Lichnerowicz-type equation

$$\Delta_L h + \mathcal{L}_X(h) = 0.$$

Moreover, $|h| \leq O(r^{-\varepsilon})$. Hence, it follows from Theorem 6.2 that $h = \lambda \text{Ric}$ for some constant $\lambda \in \mathbb{R}$. Therefore, the vector field $\hat{U} := U - V - \frac{1}{2} \lambda X$ is a Killing vector field. The relation $\mathcal{L}_{\hat{U}}(g) = 0$ implies that $\Delta \hat{U} + \text{Ric}(\hat{U}) = 0$. On the other hand, we have $\Delta \hat{U} + D_X \hat{U} = 0$ by definition of V . Thus, we conclude that $[\hat{U}, X] = \text{Ric}(\hat{U}) - D_X \hat{U} = 0$. Finally, we have

$$D^2(\mathcal{L}_{\hat{U}}(f)) = \mathcal{L}_{\hat{U}}(D^2 f) = \frac{1}{2} \mathcal{L}_{\hat{U}}(\mathcal{L}_X(g)) = \frac{1}{2} \mathcal{L}_X(\mathcal{L}_{\hat{U}}(g)) = 0.$$

Consequently, the function $\mathcal{L}_{\hat{U}}(f) = \langle \hat{U}, X \rangle$ is constant. Since X vanishes at the point where f attains its minimum, we conclude that the function $\langle \hat{U}, X \rangle$ vanishes identically. This completes the proof of Theorem 7.1.

If we apply Theorem 7.1 to the vector fields U_1, U_2, U_3 constructed in Proposition 3.3, we can draw the following conclusion:

Corollary 7.2. *We can find vector fields $\hat{U}_1, \hat{U}_2, \hat{U}_3$ on (M, g) such that $\mathcal{L}_{\hat{U}_a}(g) = 0$, $[\hat{U}_a, X] = 0$, and $\langle \hat{U}_a, X \rangle = 0$. Moreover, we have $|\hat{U}_a| \leq O(r^{\frac{1}{2}})$ and*

$$\int_{\{f=r\}} \langle \hat{U}_a, \hat{U}_b \rangle = r^2 \delta_{ab} + O(r^{2-\varepsilon})$$

if r is sufficiently large.

Theorem 1.1 is a direct consequence of Corollary 7.2.

APPENDIX A. THE EIGENVALUES OF SOME ELLIPTIC OPERATORS ON S^2

In this section, we collect some well-known results concerning the eigenvalues of certain elliptic operators on S^2 . In the following, g_{S^2} will denote the standard metric on S^2 with constant Gaussian curvature 1.

Proposition A.1. *Let σ be a one-form on S^2 satisfying*

$$\Delta_{S^2} \sigma + \mu \sigma = 0,$$

where Δ_{S^2} denotes the rough Laplacian and $\mu \in (-\infty, 1)$ is a constant. Then $\sigma = 0$.

Proof. We can find a real-valued function α and a two-form ω such that $\sigma = d\alpha + d^*\omega$. Using the Bochner formula for one-forms, we obtain

$$\begin{aligned} 0 &= \Delta_{S^2}\sigma + \mu\sigma \\ &= -dd^*\sigma - d^*d\sigma + (\mu+1)\sigma \\ &= -dd^*d\alpha - d^*dd^*\omega + (\mu+1)(d\alpha + d^*\omega) \\ &= d(\Delta_{S^2}\alpha + (\mu+1)\alpha) + d^*(\Delta_{S^2}\omega + (\mu+1)\omega). \end{aligned}$$

Consequently, the function $\Delta_{S^2}\alpha + (\mu+1)\alpha$ is constant, and the two-form $\Delta_{S^2}\omega + (\mu+1)\omega$ is a constant multiple of the volume form. Since $\mu+1 < 2$, we conclude that α is constant and ω is a constant multiple of the volume form. Thus, $\sigma = 0$, as claimed.

Proposition A.2. *Let χ be a symmetric $(0, 2)$ -tensor on S^2 satisfying*

$$\Delta_{S^2}\chi - 4\overset{\circ}{\chi} + \mu\chi = 0,$$

where $\overset{\circ}{\chi}$ denotes the trace-free part of χ and $\mu \in (-\infty, 2)$ is a constant. Then χ is a constant multiple of g_{S^2} .

Proof. The trace of χ satisfies

$$\Delta_{S^2}(\text{tr } \chi) + \mu(\text{tr } \chi) = 0.$$

Since $\mu < 2$, we conclude that $\text{tr } \chi$ is constant. Moreover, the trace-free part of χ satisfies

$$\Delta_{S^2}\overset{\circ}{\chi} + (\mu-4)\overset{\circ}{\chi} = 0.$$

Since $\mu-4 < 0$, it follows that $\overset{\circ}{\chi} = 0$. Putting these facts together, the assertion follows.

REFERENCES

- [1] G. Anderson and B. Chow, *A pinching estimate for solutions of the linearized Ricci flow system on 3-manifolds*, Calc. Var. PDE 23, 1–12 (2005)
- [2] S. Brendle, *Ricci Flow and the Sphere Theorem*, Graduate Studies in Mathematics, vol. 111, American Mathematical Society (2010)
- [3] R.L. Bryant, *Ricci flow solitons in dimension three with $SO(3)$ -symmetries*, available at www.math.duke.edu/~bryant/3DRotSymRicciSolitons.pdf
- [4] H.D. Cao, *Recent progress on Ricci solitons*, arXiv:0908.2006
- [5] H.D. Cao and Q. Chen, *On locally conformally flat gradient steady Ricci solitons*, to appear in Trans. Amer. Math. Soc.
- [6] H.D. Cao, G. Catino, Q. Chen, C. Mantegazza, and L. Mazzieri, *Bach flat gradient steady Ricci solitons*, arXiv:1107.4591
- [7] B.L. Chen, *Strong uniqueness of the Ricci flow*, J. Diff. Geom. 82, 363–382 (2009)
- [8] X.X. Chen and Y. Wang, *On four-dimensional anti-self-dual gradient Ricci solitons*, arXiv:1102.0358
- [9] H. Guo, *Area growth rate of the level surface of the potential function on the 3-dimensional steady Ricci soliton*, Proc. Amer. Math. Soc. 137, 2093–2097 (2009)

- [10] M. Gursky, *The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics*, Ann. of Math. 148, 315–337 (1998)
- [11] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. 17, 255–306 (1982)
- [12] R. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry, vol. II, 7–136, International Press, Somerville MA (1995)
- [13] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press (1973)
- [14] A. Ionescu and S. Klainerman, *On the uniqueness of smooth stationary black holes in vacuum*, Invent. Math. 175, 35–102 (2009)
- [15] A. Ionescu and S. Klainerman, *On the local extension of Killing vector-fields in Ricci flat manifolds*, arXiv:1108.3575
- [16] T. Ivey, *New examples of complete Ricci solitons*, Proc. Amer. Math. Soc. 122, 241–245 (1994)
- [17] J. Kazdan and F. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. 99, 14–47 (1974)
- [18] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arxiv:0211159
- [19] G. Perelman, *Ricci flow with surgery on three-manifolds*, arxiv:0303109
- [20] G. Perelman, *Finite extinction time for solutions to the Ricci flow on certain three-manifolds*, arxiv:0307245
- [21] L. Simon, *Isolated singularities of extrema of geometric variational problems*, In: Harmonic mapping and minimal immersions (Montecatini, 1984), Lectures Notes in Mathematics vol. 1161, 206–277 (1985)
- [22] L. Simon and B. Solomon, *Minimal hypersurfaces asymptotic to quadratic cones in \mathbb{R}^{n+1}* , Invent. Math. 86, 535–551 (1986)
- [23] M. Struwe, *Curvature flows on surfaces*, Ann. Scuola Norm. Sup. Pisa Serie V, 1, 247–274 (2002)
- [24] P. Topping, *Lectures on the Ricci Flow*, London Mathematical Society Lecture Notes Series, vol. 325, Cambridge University Press, Cambridge (2006)
- [25] X.J. Wang, *Convex solutions to the mean curvature flow*, Ann. of Math. 173, 1185–1239 (2011)
- [26] Z.H. Zhang, *On the completeness of gradient Ricci solitons*, Proc. Amer. Math. Soc. 137, 2755–2759 (2009)

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA 94305